数理解析研究所講究録
540巻 1984年 85-90

ON THEORIES HAVING A FINITE NUMBER OF NON-ISOMORPHIC COUNTABLE MODELS

Akito TSUBOI (坪井明人)
University of Tsukuba

§0. Introduction. In this paper we shall state some interesting facts concerning non-ω-categorical theories which have only finitely many countable models. Although many examples of such theories are known, almost all of them are essentially the same in the following sense: they are obtained from ω-categorical theories, called base theories below, by adding axioms for infinitely many constant symbols. Moreover all known base theories have the (strict) order property in the sense of [6], and so they are unstable. For example, well-known Ehrenfeucht's example which have three countable models has the theory of dense linear order as its base theory.

Many papers including [4] and [5] are motivated by the conjecture that every non-ω-categorical theory with a finite number of countable models has the (strict) order property, but this conjecture still remains open. (Of course there are partial positive solutions. For example, in [4], Pillay showed that if such a theory has few links (see [1]), then it has the strict order property.) In this paper we prove the instability of the base theory T_0 of such a theory T rather than T itself.

Our main theorem is a strengthning of the following which is also our result: if a theory T_0 is stable and -categorical, then T_0 cannot be extended to a theory T which has n countable models $(1 < n < \omega)$, by adding axioms for new constants.

<u>§1. Preliminaries.</u> Our notaions and conventions are fairly standard. T, T_n $(n<\omega)$ will denote complete theories formulated in some countable languages. M, M_n $(n<\omega)$ will denote countable models of such theories. $\bar{a}$, $\bar{b}$,... will be used to denote finite sequences of elements is some models. Types are complete types without parameters, and will be denoted by p, q,$\cdots$. $I(\omega,T)$ is the number of countable models of T.

<u>Definition 1.</u> A type $q(\bar{x},\bar{y})$ is said to be an order expression if $q(\bar{x},\bar{y})$ is principal over the first variables $\bar{x}$ and non-principal over the second variables $\bar{y}$.

<u>Definition 2 (Benda).</u> A type $p(\bar{x})$ is said to be a powerful type of T if every model of T which realizes it realizes every type $q(\bar{y}) \in S(T)$.

We now state some facts which are necessary for proving our results.

<u>Fact (i).</u> If $I(\omega,T) < \omega$, then a powerful type $p(\bar{x})$ of T exists.

<u>Fact (ii).</u> Let $1 < I(\omega,T) < \omega$ and $p(\bar{x}), q(\bar{y}) \in S(T)$. If $p(\bar{x})$ is a powerful type of T, then there is an order expression $r(\bar{x},\bar{y})$ which extends $p(\bar{x})$ and $q(\bar{y})$.

<u>Fact (iii).</u> A theory T is ω-categorical if and only if it has only finitely many non-equivalent formulas $\varphi(\bar{x})$, for each $\bar{x}$.

Fact (i) and Fact (ii) can be obtained by easy observations (see [4] for reference). Fact (iii) can be seen, e.g. in [2].

<u>§2. Main theorem and its corollary.</u> We prove the following theorem which will show the difficulty in constructing a stable theory T with $1 < I(\omega,T) < \omega$, even if such a theory exists.

<u>Theorem.</u> Let T_i $(i<\omega)$ and T be theories with the following properties:

(i) $T_i \subseteq T_{i+1}$ for all $i<\omega$; $T=\bigcup_{i<\omega} T_i$;
(ii) T_i is ω-categorical for all $i<\omega$;
(iii) $1 < I(\omega,T) < \omega$.

Then T has the order property, and so T is unstable. (So some T_i is unstable.)

First we prove the following lemma:

<u>Lemma.</u> Let $p(\bar{x})$ be a powerful type of T with $1 < I(\omega,T) < \omega$ and $q(\bar{x},\bar{y})$ an order expression which extends $p(\bar{x}) \cup p(\bar{y})$. Then there is a sequence $\{\bar{a}_i\}_{i<\omega}$ s.t. $tp(\bar{a}_i,\bar{a}_{i+1}) = q(\bar{x},\bar{y})$ for all $i<\omega$, and $tp(\bar{a}_i,\bar{a}_j)$ is an order expression iff $i<j<\omega$.

<u>Proof.</u> We construct two sequences $\{\bar{a}_i\}_{i<\omega}$ of realizations of p and $\{M_i\}_{i<\omega}$ of models of T such that for each $i<\omega$,

(1) $M_i \succ M_{i+1}$;

(2) M_i is prime over $\bar{a}_i$ (hence $\bar{a}_i \in M_i$);

(3) $tp(\bar{a}_i,\bar{a}_{i+1}) = q(\bar{x},\bar{y})$.

This is done inductively. Let $\bar{a}_0$ be a realization of p and M_0 a prime model over $\bar{a}_0$. (M_0 exists since $I(\omega,T)<\omega$.) Assume that we have already defined $\{\bar{a}_i\}_{i<n}$ and $\{M_i\}_{i<n}$. Since M_{n-1} is prime over $\bar{a}_{n-1}$ and $q(\bar{x},\bar{y})$ is an order expression, we can choose $\bar{a}_{n-1} \in M_{n-1}$ such that $tp(\bar{a}_{n-1},\bar{a}_n) = q(\bar{x},\bar{y})$. Let $M_n \prec M_{n-1}$ be a prime model over $\bar{a}_n$. It is then clear that (1) - (3) are satisfied by $\{\bar{a}_i\}_{i\leqq n}$ and $\{M_i\}_{i\leqq n}$. Thus the construction can be carried out. We prove that $\bar{a}_{i\ i<}$ has the desired properties. It is sufficient to prove that $tp(\bar{a}_i,\bar{a}_j)$ is an order expression if $i < j$. Let $i < j$. Then clearly $q_{i,j}(\bar{x},\bar{y}) = tp(\bar{a}_i,\bar{a}_j)$ is principal over $p(\bar{x})$. So we only have to show that $q_{i,j}(\bar{x},\bar{y})$ is non-principal over $p(\bar{y})$. But this is clear, since $\bar{a}_{j-1}$ is prime over $\bar{a}_i{}^\frown\bar{a}_j$ and $tp(\bar{a}_{j-1},\bar{a}_j)$ is an order expression.

Proof of Theorem. By Fact (i) and Fact (ii), we can choose a powerful type $p(\bar{x})$ and an order expression $q(\bar{x},\bar{y})$ which extends $p(\bar{x}) \cup p(\bar{y})$. So by Lemma, there is a sequence $\{\bar{a}_i\}_{i<\omega}$ M_0 of realizations of $p(\bar{x})$ such that all $tp(\bar{a}_i,\bar{a}_j)$ $(i<j<\omega)$ are order expressions and all $\bar{a}_i{}^\frown\bar{a}_{i+1}$ $(i<\omega)$ realizes the same type $q(\bar{x},\bar{y})$. Choose a number $m<\omega$ and a formula $\varphi(\bar{x},\bar{y}) \in L(T_m)$ such that $p(\bar{x}) \cup \{\varphi(\bar{x},\bar{y})\}$ proves $q(\bar{x},\bar{y})$. For each $i<\omega$, let $\varphi_i(\bar{x},\bar{y})$ be the formula $\exists \bar{x}_0,\ldots,\bar{x}_{i-1}[\varphi(\bar{x},\bar{x}_0) \wedge \varphi(\bar{x}_0,\bar{x}_1) \wedge \cdots \wedge \varphi(\bar{x}_{i-1},\bar{y})]$. Since each φ_i is an $L(T_m)$-formula and T_m is ω-categorical, by Fact (iii), there are only finitely many non-equivalent formulas in $\{\varphi_i\}_{i<\omega}$. So $\psi(\bar{x},\bar{y}) = \bigvee_{i<\omega} \varphi_i(\bar{x},\bar{y})$ is a first order formula. Now it is a routine to check that, for all $i,j<\omega$, $\psi(\bar{a}_i,\bar{a}_j)$ holds in M_0 iff $i < j$. Thus T has the order property.

Corollary. Let T_0 be stable and ω-categorical. Let T be an extension of T_0 obtained by the addition of axioms for new constant symbols. Then $I(\omega,T) = 1$ or $I(\omega,T) \geq \omega$.

References.

[1] M. Benda, Remarks on countable models, Fundamenta Mathematicae, vol. 81 (1974), pp. 107-119.

[2] C.C. Chang and H.J. Keisler, Model theeory, North-Holland, Amsterdam, 1973.

[3] A. Pillay, Number of countable models, J.S.L., vol. 43 (1978), pp. 492-496.

[4] A. Pillay, Instability and theories with few models, Proceedings of the American Mathematical Society, vol 80 (1980), pp. 461-468.

[5] A. Pillay, Stable theories, pseudoplanes and the number of countable models, to appear.

[6] S. Shelah, Classification theory and the number of non-isomorphic models, North-Holland, Amsterdam, 1978.

[7] A. Tsuboi, On theories having a finite number of non-isomorphic countable models, to appear in J.S.L.

[8] R.E. Woodrow, A note on countable models, J.S.L., vol.41 (1976), pp. 672-680.

[9] R.E. Woodrow, Theories with a finite number of countable models, J.S.L., vol. 43 (1978), pp. 442-455.